Mathematical Music

by

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THE concept of a chord as a combination of notes is very familiar but its general structure is perhaps not so well known. Thus we associate with the dominant C-major chord C E G C its inversions E G C E and G C E G, and with each of the twelve notes we associate such a dominant major chord corresponding to each of the major scales. In other words, this "chord" can be considered as a fundamental musical element irrespective of the scale or particular inversion.

It will be seen that the characteristic of the chord is merely the sequence of intervals 4 3 5, adding to 12, the inversions corresponding to the cyclic permutations 3 5 4 and 5 4 3. Such a splitting of the number 12 is an ordinary partition of 12 into three parts, ignoring the order of the numbers, or a composition of 12, taking order into account.

We therefore define the general chord as a composition of the number 12, together with the cyclic permutations of the composition, the characteristic type of the chord being the partition given by the numbers contained in the composition. It is clear that, corresponding to any chord of m notes, there is a chord of $12 - m$ notes obtained from all the notes not used in the original chord. This will be called the "conjugate" chord. For example, the conjugate chord of the dominant minor chord (3 4 5) of three notes, is (1 2 1 1 2 1 1 1 2) of nine notes, partitional type $(2^3\ 1^6)$. (It will be noticed that the minor and major dominant chords are the only two of the partitional type (5 4 3).) Because of the above, to enumerate all the possible chords of 12 notes, we need only consider those of 6 or less.

For chords with total interval greater than twelve, we replace successively the highest notes of the chord by the equivalent notes within the first octave after the first note of the chord. Thus the chords C G D′ A′, C G D″ A″, etc., are all equivalent to C D G A.

The following is an enumeration of all the chords of six or less notes, according to their partitional type.

1 *note*	1 chord		of type (12).
2 *notes*	6 chords,	1 each	of type (11 1) (10 2) (9 3) (8 4) (7 5) (6 6).
3 *notes*	14 chords,	2 each	of type (9 2 1) (8 3 1) (7 4 1) (7 3 2) (6 5 1) (6 4 2) (5 4 3).
	4 chords,	1 each	of type $(10\ 1^2)$ $(8\ 2^2)$ $(6\ 3^2)$ $(5^2\ 2)$.
	1 chord		of type (4^3).
Total	19		
4 *notes*	12 chords,	6 each	of type (6 3 2 1) (5 4 2 1).
	24 chords,	3 each	of type $(8\ 2\ 1^2)$ $(7\ 3\ 1^2)$ $(7\ 2^2\ 1)$ $(6\ 4\ 1^2)$ $(5\ 3^2\ 1)$ $(5\ 3\ 2^2)$ $(4^2\ 3\ 1)$ $(4\ 3^2\ 2)$.
	4 chords,	2 each	of type $(5^2\ 1^2)$ $(4^2\ 2^2)$.
	2 chords,	1 each	of type $(9\ 1^3)$ $(6\ 2^3)$.
	1 chord		of type (3^4).
Total	43		
5 *notes*	24 chords,	12 each	of type $(5\ 3\ 2\ 1^2)$ $(4\ 3\ 2^2\ 1)$.
	18 chords,	6 each	of type $(6\ 2^2\ 1^2)$ $(4^2\ 2\ 1^2)$ $(4\ 3^2\ 1^2)$.
	20 chords,	4 each	of type $(7\ 2\ 1^3)$ $(6\ 3\ 1^3)$ $(5\ 4\ 1^3)$ $(5\ 2^3\ 1)$ $(3^3\ 2\ 1)$.
	2 chords		of type $(3^2\ 2^3)$.
	2 chords,	1 each	of type $(8\ 1^4)$ $(4\ 2^4)$.
Total	66		
6 *notes*	20 chords		of type $(4\ 3\ 2\ 1^3)$.
	16 chords		of type $(3^2\ 2^2\ 1^2)$.
	20 chords,	10 each	of type $(5\ 2^2\ 1^3)$ $(4\ 2^3\ 1^2)$.
	15 chords,	5 each	of type $(6\ 2\ 1^4)$ $(5\ 3\ 1^4)$ $(3\ 2^4\ 1)$.
	4 chords		of type $(3^3\ 1^3)$.
	3 chords		of type $(4^2\ 1^4)$.
	1 chord		of type $(7\ 1^5)$.
	1 chord		of type (2^6).
Total	80		

In the above table, the partitions corresponding to chords of m notes are all the partitions of 12 containing m parts. It will be noticed that the number of chords of given partitional type depends only on the sub-partition formed by the indices. Thus, $(7\ 2\ 1^3)$ has sub-partition $(3\ 1^2)$ and all the partitions with this sub-partition 5 yield 4 chords of ~~4~~ notes.

This is an example of the endless scope which exists for mathematicians to formalise and generalise the rules of harmony. A major scale, for instance, can be identified with the chord (2 2 1 2 2 2 1), the conjugate being (2 3 2 2 3), and it is not difficult to formulate rules for operating on these chords in many ways. For example, a chord can be contracted to one of fewer notes, simply by adding together any number of consecutive intervals, arpeggios may be indicated by the consecutive playing of the cyclic forms of a given chord, or the transition from major to minor chord can be indicated by the permutation of the intervals (3 4) in (4 3 5). After all, the whole of music consists merely of all the possible combinations of the twelve notes (or more if we use micro-tones) amongst the different instruments and octaves! It thus lends itself to a systematic classification of the kind considered here. However, I feel confident that music will remain the creative art of the musician rather than the mathematician.

I am indebted to Roberto Gerhard for suggesting to me the notion of the general chord, and the problem of enumerating such chords.

International Congress of Mathematicians

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THIS will take place from 14th to 21st August. A number of mathematicians will deliver one-hour and half-hour addresses; there will also be daily sessions for fifteen-minute communications. The Congress is divided into eight sections according to subjects. A programme of entertainments and excursions is planned. Those who wish to receive further information should communicate direct with the Secretary of the Congress:

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